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EXCEPTIONAL DIVISORS WHICH ARE NOT UNIRULED BELONG TO THE IMAGE OF THE NASH MAP

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Dedicated to H. Hironaka

Abstract. We prove that, if X is a variety over an uncountable algebraically closed field k of characteristic zero, then any irreducible exceptional divisor E on a resolution of singularities of X which is not uniruled, belongs to the image of the Nash map, i.e. corresponds to an irreducible component of the space of arcs X_∞^{Sing} on X centered in Sing X. This reduces the Nash problem of arcs to understanding which uniruled essential divisors are in the image of the Nash map, more generally, how to determine the uniruled essential divisors from the space of arcs.

1. Introduction

In the midsixties, J. Nash ([Na]) initiated the study of the space of arcs X_∞ of a singular variety X to understand what the various resolutions of singularities of X have in common. His work was developed in the context of the proof of resolution of singularities in characteristic zero by H. Hironaka ([Hi]). From the existence of a resolution of singularities, Nash deduces that the space of arcs X_∞^{Sing} on X centered in the singular locus Sing X of X, has a finite number of irreducible components. More precisely, he defines an injective map $\mathcal{N}_X$, now called the *Nash map*, from the set of irreducible components of X_∞^{Sing} which are not contained in (Sing $X)_\infty$ to the set of *essential divisors over* X, i.e. exceptional irreducible divisors which appear up to birational equivalence on every resolution of singularities of X. He asks whether this map is surjective, or more generally, how complete is the description of the essential divisors by the image of the Nash map.

In 1980, the first author ([Le]) proposed to approach the above problem using arcs in the space of arcs X_∞, or equivalently, wedges. However, the "curve selection lemma" does not hold in X_∞ because it is not a Noetherian space. This obstacle was shortcut in 2006 by the second author ([Re2]), by introducing the class of *generically stable irreducible subsets of* X_∞, and proving a curve selection lemma for the corresponding *stable points of* X_∞. The residue field of such points being a transcendental extension of k of infinite transcendence degree, the stable points are very far from being closed points. In [Re2] the image of the Nash map is characterized in terms of a property of lifting wedges centered at certain stable points to some resolution of singularities of X. In this paper, given an essential divisor ν over X, we introduce a property of *lifting wedges centered* at enough closed points of a locally closed subset of X_∞ associated to ν, which implies that ν belongs to

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the image of the Nash map $\mathcal{N}_X$ (see 2.1, def. 2.10 and cor. 2.15).

In 2003, S. Ishii and J. Kollar ([IK]) proved that the surjectivity of the Nash map fails in general for $\dim X \geq 4$. The fact that unirational does not imply rational for projective varieties of dimension ≥ 3 is crucial in the construction of their example. The smooth cubic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ is the first known example of a unirational variety which is not rational, and as a consequence, it is uniruled but not birationally ruled. In fact, if k is algebraic closed, given a smooth hypersurface E in $\mathbb{P}^d_k$ which is uniruled but not birationally ruled, singular varieties X of dimension $d \geq 4$ such that E is an essential divisor over X which is not in the image of the Nash map are constructed in [IK].

In section 3, we prove that, if the base field k is algebraically closed of characteristic zero and uncountable, then any irreducible exceptional divisor E on a resolution of singularities Y of X which is not uniruled, is an essential divisor over X which belongs to the image of the Nash map (thm. 3.3). Using cor. 2.15, this follows from Luroth's theorem by looking at the elimination of the points of indeterminacy of the rational maps to Y coming from the wedges. This reduces the Nash problem for surfaces to decide which rational curves on the minimal desingularization belong to the image of the Nash map. Finally, in the appendix (prop. 4.2), we show that the surjectivity of the Nash map for normal surface singularities over the field $\mathbb{C}$ of complex numbers would follow from proving that every quasirational surface singularity over $\mathbb{C}$ has a resolution which enjoys the property of lifting wedges with respect to each essential divisor. This makes more significant studying rational surface singularities ([Le], [Re1], [LR]).

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2. Wedges and the image of the Nash map

2.1. We begin this section by a brief introduction to the spaces of arcs and wedges. For more details on arcs, see [DL], [EM], [IK], [Vo].

Let k be a field and let X be a k-scheme. Given a field extension $k \subseteq K$, a K-*arc* on X is a k-morphism Spec $K[[t]] \to X$. The K-arcs on X are the K-rational points of a k-scheme X_∞ called the *space of arcs* of X. More precisely, $X_\infty = \lim_{\leftarrow} X_n$, where, for $n \in \mathbb{N}$, X_n is the k-scheme of n-jets whose K-rational points are the k-morphisms Spec $K[t]/(t)^{n+1} \to X$. The projective limit $\lim_{\leftarrow} X_n$ exists in the category of k-schemes, because, for $n' \geq n$, the natural projections $X_{n'} \to X_n$ are affine morphisms. For every k-algebra A, we have a natural isomorphism

$$Hom_k(\text{Spec } A,\ X_\infty) \cong Hom_k(\text{Spec } A[[t]],\ X). \tag{1}$$

Given a K-arc h : Spec $K[[t]] \to X$, we will call the image $h(O)$ in X of the closed point O of Spec $K[[t]]$ the *center* of h. This is also the image by the natural projection $j_0 : X_\infty \to X$ of the point of X_∞ induced by the K-rational point of X_∞ corresponding to h by (1). For $P \in X_\infty$ with residue field $\kappa(P)$, we will denote by h_P the $\kappa(P)$-arc on X corresponding by (1) to the $\kappa(P)$-rational point of X_∞ induced by P. We will also call the center $j_0(P)$ of h_P the center of P. Given $P \in X_\infty$ as above with center P_0 on X, we will denote by ν_P the *order function* $\text{ord}_t h^\sharp_P : \mathcal{O}_{X,P_0} \to \mathbb{N} \cup \{\infty\}$ with $h^\sharp_P : \mathcal{O}_{X,P_0} \to \kappa(P)[[t]]$ induced by h_P.

A K-*wedge* on X is a k-morphism Spec $K[[\xi,t]]\to X$. The K-wedges on X are the K-rational points of a k-scheme $X_{\infty,\infty}$. We will call $X_{\infty,\infty}$ the *space of wedges* of X. Since by (1), for every k-algebra A, we have the natural isomorphisms
(2)
$$Hom_k(\text{Spec } A,\ (X_\infty)_\infty)\cong Hom_k(\text{Spec } A[[\xi]],\ X_\infty)\cong Hom_k(\text{Spec } A[[\xi,t]],\ X)$$

the space of wedges $X_{\infty,\infty}$ and the space of arcs $(X_\infty)_\infty$ of X_∞ are naturally isomorphic k-schemes, and we have a natural isomorphism

$$(2') \qquad Hom_k(\text{Spec } A,\ X_{\infty,\infty})\cong Hom_k(\text{Spec } A[[\xi,t]],\ X).$$

The k-scheme $X_{\infty,\infty}$ is the projective limit of the k-schemes $\{X_{r,n}\}_{(r,n)\in\mathbb{N}^2}$, where $X_{r,n}$ is the k-scheme of r-jets of X_n, whose K-rational points are the k-morphisms Spec $K[\xi,t]/(\xi^{r+1},t^{n+1})\to X$, with the natural affine transition morphisms $X_{r',n'}\to X_{r,n}$, for $r'\geq r$ and $n'\geq n$.

Given a K-wedge $\Phi:\text{Spec } K[[\xi,t]]\to X$, we call the image in X_∞ of the closed point (resp. generic point) of Spec $K[[\xi]]$ by the K-arc $h_\Phi:\text{Spec } K[[\xi]]\to X_\infty$ corresponding to Φ by (2), the *special arc* (resp. *generic arc*) of Φ. With the above terminology, the special arc of Φ is nothing but the center of the arc h_Φ. This is also the image by the natural projection $j_{0,\infty}:X_{\infty,\infty}=\lim_{\leftarrow}X_{r,n}\to X_\infty=\lim_{\leftarrow}X_{0,n}$ of the point of $X_{\infty,\infty}$ induced by the K-rational point of $X_{\infty,\infty}$ corresponding to Φ by (2′). Thus, we will also say that Φ is *centered* at $P\in X_\infty$ to mean that P is the special arc of Φ.

For $\mathbf{P}\in X_{\infty,\infty}$ with residue field $\kappa(\mathbf{P})$, we will denote by $\Phi_{\mathbf{P}}$ the $\kappa(\mathbf{P})$-wedge on X corresponding by (2′) to the $\kappa(\mathbf{P})$-rational point of $X_{\infty,\infty}$ induced by $\mathbf{P}$.

A K-wedge Φ on X can also be viewed as a $K[[t]]$-point of $X_\infty=\lim_{\leftarrow}X_{r,0}$. We will call the image in X_∞ of the closed point of Spec $K[[t]]$ the *arc of centers* of Φ. This is also the image by the natural projection $j_{\infty,0}:X_{\infty,\infty}=\lim_{\leftarrow}X_{r,n}\to X_\infty=\lim_{\leftarrow}X_{r,0}$ of the point of $X_{\infty,\infty}$ induced by the K-rational point of $X_{\infty,\infty}$ corresponding to Φ by (2′).

If X is a variety over k, i.e. a reduced and irreducible separated k-scheme of finite type, we will denote by X_∞^{Sing} the closed set $j_0^{-1}(\text{Sing } X)$ of X_∞ consisting of the arcs P centered at some singular point of X. Finally, we will denote by $X_{\infty,\infty}^{\text{Sing}}$ the closed set $j_{\infty,0}^{-1}((\text{Sing } X)_\infty)$ consisting of the wedges $\mathbf{P}$ such that the generic arc of $\Phi_{\mathbf{P}}$ belongs to X_∞^{Sing}.

Note that

$$(\mathbb{A}_k^m)_\infty=\text{Spec } k[\{\underline{X}_n\}_{n\geq 0}] \qquad (\mathbb{A}_k^m)_{\infty,\infty}=\text{Spec } k[\{\underline{X}_{r,n}\}_{r,n\geq 0}]$$

where, $\underline{X}_n=(X_{1,n},\ldots,X_{m,n})$ for $n\geq 0$, and $\underline{X}_{r,n}=(X_{1,r,n},\ldots,X_{m,r,n})$, for $r,n\geq 0$. Let us consider the morphism of k-algebras $\mathcal{O}(\mathbb{A}_k^m)\to\mathcal{O}((\mathbb{A}_k^m)_\infty)[[t]]$ induced in (1) by the identity map in $(\mathbb{A}_k^m)_\infty$. For any $l\in\mathcal{O}(\mathbb{A}_k^m)$, the image of l in $\mathcal{O}((\mathbb{A}_k^m)_\infty)[[t]]$ by the previous morphism will be denoted by

$$(3) \qquad \sum_{n=0}^{\infty} L_n\ t^n\in\mathcal{O}((\mathbb{A}_k^m)_\infty)[[t]].$$

Analogously, for $l\in\mathcal{O}(\mathbb{A}_k^m)$, the image of l in $\mathcal{O}((\mathbb{A}_k^m)_{\infty,\infty})[[\xi,t]]$ by the morphism of k-algebras $\mathcal{O}(\mathbb{A}_k^m)\to\mathcal{O}((\mathbb{A}_k^m)_{\infty,\infty})[[\xi,t]]$ induced in (2′) by the identity map in

$(\mathbb{A}_k^m)_{\infty,\infty}$ will be denoted by

$$\sum_{r,n\geq 0} L_{r,n}\ \xi^r\ t^n \in \mathcal{O}((\mathbb{A}_k^m)_{\infty,\infty})[[\xi,t]]. \tag{4}$$

If X is a closed subscheme of $\mathbb{A}_k^m$, and $I_X \subset k[x_1,\dots,x_m]$ is the ideal defining X in $\mathbb{A}_k^m$, then we have

$$X_\infty = \text{Spec } k[\{\underline{X}_n,\}_{n\geq 0}]\ /\ (\{F_n\}_{n\geq 0, f\in I_X})$$

$$X_{\infty,\infty} = \text{Spec } k[\{\underline{X}_{r,n}\}_{r,n\geq 0}]\ /\ (\{F_{r,n}\}_{r,n\geq 0, f\in I_X})$$

We will use the same symbol to denote the element of $k[x_1,\dots,x_m]$, (resp. $\mathcal{O}((\mathbb{A}_k^m)_\infty)$), (resp. $\mathcal{O}((\mathbb{A}_k^m)_{\infty,\infty})$) and its class in $\mathcal{O}(X)$, (resp. $\mathcal{O}(X_\infty)$), (resp. $\mathcal{O}(X_{\infty,\infty})$).

2.2. From now on, *k will denote an uncountable field of characteristic zero*, and X a variety over k of dimension d. Let $p : Y \to X$ be a resolution of singularities of X, i.e. a proper, birational morphism, with Y nonsingular, such that the induced morphism $Y \setminus p^{-1}(Sing\ X) \to X \setminus Sing\ X$ is an isomorphism. Given an irreducible component E of the exceptional locus $p^{-1}(Sing\ X)$ of p of codimension one, for short an *exceptional divisor* from now on, we will denote by Y_∞^E the inverse image of E by the natural projection $j_0^Y : Y_\infty \to Y$. Given a dense open subset U of E in the nonsingular locus of $p^{-1}(Sing\ X)$, we will denote by $Y_\infty^\dagger(U)$ the set of points Q in Y_∞ such that $j_0^Y(Q) \in U$ and such that the corresponding arc h_Q intersects E transversally. Since Y is nonsingular, Y_∞^E and $Y_\infty^\dagger(U)$ are reduced and irreducible. Moreover, $Y_\infty^\dagger(U)$ is open in Y_∞^E.

Let $p_\infty : Y_\infty \to X_\infty$ be the morphism induced by p. The closure N_E of $p_\infty(Y_\infty^E)$ is an irreducible subset of X_∞^{Sing}. Note that, if P_E denotes the generic point of N_E, the order function ν_{P_E} coincides with the restriction of the divisorial valuation ν_E defined by E to the local ring of X at the generic point of $p(E)$, and that, for every affine open subset V of X such that $V \cap p(E) \neq \emptyset$, and for every $g \in \mathcal{O}_X(V)$, we have

$$\nu_{P_E}(g) = inf_{P\in N_E}\nu_P(g).$$

For U as above, we set $N^\dagger(U) := p_\infty(Y_\infty^\dagger(U))$.

Lemma 2.3. *Suppose in addition that p is the blowing-up of a subscheme D whose associated reduced scheme D_{red} is $Sing\ X$, for short an Hironaka resolution of singularities (see [Hi] I p. 132). There exists a dense open subset U of E as above such that $N^\dagger(U)$ is open in N_E ; moreover, the morphism $p_\infty : Y_\infty \to X_\infty$ induces an isomorphism of schemes $Y_\infty^\dagger(U) \to N^\dagger(U)$.*

Proof : We may assume X to be affine. Let $I = (g_0,\dots,g_s)\mathcal{O}(X)$ be the ideal defining D in X. We have $n_i := \nu_E(g_i) \geq 1$, for $0 \leq i \leq s$; for simplicity, suppose that $n_0 = inf\{n_i\}_{0\leq i\leq s}$.

Let $\Omega_0 = \text{Spec } \mathcal{O}(X)[\frac{g_1}{g_0},\dots,\frac{g_s}{g_0}]$ be the open subset of Y on which $I\mathcal{O}(\Omega_0) = (g_0)\mathcal{O}(\Omega_0)$, and let U be the subset of $\Omega_0 \cap E$ consisting of the nonsingular points of the reduced exceptional locus of p. We will show that, for $P \in N_E$, we have $P \in N^\dagger(U)$ if and only if $\nu_P(g_0) = n_0$.

First note that, since $(div\ g_0)_U = n_0\ E\cap U$, for $P \in N^\dagger(U)$ we have $\nu_P(g_0) = n_0$. To prove the converse, we first observe that, since for $P \in N_E$, we have $\nu_P(g_0) \geq n_0$, the set $\{P \in N_E\ /\ \nu_P(g_0) = n_0\}$ is the affine open set of N_E where $(G_0)_{n_0} \neq 0$. Moreover, since in addition, for $P \in N_E$, we have $\nu_P(g_i) \geq \nu_{P_E}(g_i) = n_i \geq n_0$, for $0 \leq i \leq s$, the natural morphism Spec $\mathcal{O}(N_E)_{(G_0)_{n_0}}[[t]] \to X$ corresponding to the inclusion $N_E \setminus V((G_0)_{n_0}) \hookrightarrow X_\infty$ by (1), lifts to $\Omega_0 \subset Y$. As a consequence, the map

sending $P \in N_E \setminus V((G_0)_{n_0})$ to the center of the lifting $\widetilde{h}_P$ to Y of the arc h_P, is a morphism of schemes. Since this map sends P_E to the generic point of E, its image is contained in $E \cap \Omega_0$. We conclude that, for any $P \in N_E \setminus V((G_0)_{n_0})$, the arc $\widetilde{h}_P$ intersects transversally E at a point in Ω_0 which is nonsingular in $p^{-1}(Sing\ X)$; so $N_E \setminus V((G_0)_{n_0}) = N^\dagger(U)$.

Finally, the isomorphism $Y^\dagger_\infty(U) \to N^\dagger(U)$ follows immediately, by (1) again, from the lifting Spec $\mathcal{O}(N_E)_{(G_0)_{n_0}}[[t]] \to Y$.

Remark 2.4. In the previous lemma, we have used that, for a ring R, a series S in $R[[t]]$ is a unit if and only if its constant term $S(0)$ is invertible in R. In the next lemma, we will use the following straightforward observation (which provides us with an algebraic formula computing the coefficients in t of the formal power series $y_i(t) = \frac{g_i(\underline{x}(t))}{g_0(\underline{x}(t))}$ from the coefficients of $\underline{x}(t)$) :

There exist polynomials $P_n \in \mathbb{Z}[U_{n_0}, V_{n_0}, \ldots, U_{n_0+n}, V_{n_0+n}]$, for $n \geq 0$, such that the equality

$$\left(\sum_{n\geq 0} Y_n\ t^n\right)\ \left(\sum_{n\geq n_0} U_n\ t^n\right) = \left(\sum_{n\geq n_0} V_n\ t^n\right)$$

in $k\left[\{Y_n\}_{n\geq 0}, \{U_n, V_n\}_{n\geq n_0}\right][[t]]$, is equivalent to

$$Y_n\ U_{n_0}^{n+1} = P_n(U_{n_0}, V_{n_0}, \ldots, U_{n_0+n}, V_{n_0+n}), \qquad \text{for } n \geq 0. \tag{5}$$

Lemma 2.5. *Let $p : Y \to X$ be an Hironaka resolution of singularities of X. Let $k \subseteq K$ be a field extension, and let Φ be a K-wedge whose special arc is centered in $Sing\ X$, but does not factor through $Sing\ X$.*

If Φ does not lift to Y, then there exists a locally closed subset Λ of $X_{\infty,\infty}$ such that Φ is a K-point of Λ, and such that, for any field extension $k \subseteq L$, any L-point Ψ of Λ is a L-wedge on X which does not lift to Y.

Proof : As in the proof of lemma 2.3, we may assume that X is affine and that p is the blowing-up of the ideal $I = (g_0, \ldots, g_s)\mathcal{O}(X)$ with $V(I) = Sing\ X$.

First note that the special arc of any L-wedge Ψ on X is centered in $Sing\ X$, but does not factor through $Sing\ X$, if and only if the ideal $\Psi^\sharp(I)(0,t)$ in $L[[t]]$, image of $\Psi^\sharp(I)$ by the map $L[[\xi,t]] \to L[[t]]$, $\xi \mapsto 0, t \mapsto t$, is neither (1) nor (0), or equivalently, $0 < ord_t \Psi^\sharp(I)(0,t) < \infty$. Moreover, such a Ψ lifts to Y if and only if there exists i_0, $0 \leq i_0 \leq s$, with $ord_t \Psi^\sharp(I)(0,t) = ord_t \Psi^\sharp(g_{i_0})(0,t)$, and such that the ideal generated by $\Psi^\sharp(I)$ in $L[[\xi,t]]$ is the principal ideal $(\Psi^\sharp\,(g_{i_0}))$.

For simplicity suppose that $m_0 := ord_t \Phi^\sharp(g_0)(0,t) = ord_t \Phi^\sharp(I)(0,t)$. Then $0 < m_0 < \infty$. The L-wedges Ψ such that $ord_t \Psi^\sharp(g_i)(0,t) \geq m_0$ for $0 \leq i \leq s$, and $ord_t \Psi^\sharp(g_0)(0,t) = m_0$, are the L-points of a locally closed subset $\Lambda_{m_0} := \{(G_i)_{0,n} = 0,\ 0 \leq i \leq s,\ 0 \leq n < m_0,\ (G_0)_{0,m_0} \neq 0\}$. Any L-point of Λ_{m_0} is a L-wedge which lifts to Y if and only if $\left(\Psi^\sharp(I)\right) = \left(\Psi^\sharp(g_0)\right)$, and Φ is a K-point of Λ_{m_0}.

Now suppose that Φ does not lift to Y, so there exists i_1, $1 \leq i_1 \leq s$, such that $\Phi^\sharp(g_{i_1}) \notin \left(\Phi^\sharp(g_0)\right)$. For simplicity, suppose that $i_1 = 1$. Since $ord_t \Phi^\sharp(g_0)(0,t) = m_0 < \infty$, we have that $n_0 := ord_t \Phi^\sharp(g_0) \leq m_0$, hence $e_0 := ord_\xi \left(G_0^\Phi\right)_{n_0} < \infty$, where $\Phi^\sharp(g_0) = \sum_{n\geq n_0} \left(G_0^\Phi\right)_n t^n$ in $K[[\xi]][[t]]$. So either condition (i) or, in view of (5), condition (ii) below holds :

(i) $n_1 := ord_t \Phi^\sharp(g_1) < n_0$,

(ii) $ord_t \Phi^\sharp(g_1) \geq n_0$ and there exists $n \geq n_0$ such that

$$e(n) := ord_\xi P_n\left((G_0^\Phi)_{n_0}, (G_1^\Phi)_{n_0}, \ldots, (G_0^\Phi)_{n_0+n}, (G_1^\Phi)_{n_0+n}\right) < (n+1)e_0,$$

where $\Phi^\sharp(g_1) = \sum_{n \geq n_0} (G_1^\Phi)_n t^n$ in $K[[\xi]][[t]]$.

In both cases, the L-wedges Ψ such that $ord_t \Psi^\sharp(g_0) = n_0$ and $ord_\xi \left(G_0^\Psi\right)_{n_0} = e_0$ are the L-points of a locally closed subset $\Lambda^0_{n_0,e_0} := \{(G_0)_{r,n} = 0$ for $0 \leq n < n_0$, $(G_0)_{r,n_0} = 0$ for $0 \leq r < e_0$, $(G_0)_{e_0,n_0} \neq 0\}$ in $X_{\infty\infty}$, and Φ is a K-point of $\Lambda^0_{n_0,e_0}$.

In case (i), we have $e_1 := ord_\xi \left(G_1^\Phi\right)_{n_1} < \infty$. Similarly, the L-wedges Ψ such that $ord_t \Psi^\sharp(g_1) = n_1$ and $ord_\xi \left(G_1^\Psi\right)_{n_1} = e_1$ are the L-points of a locally closed subset $\Lambda^1_{n_1,e_1}$ in $X_{\infty\infty}$, and Φ is a K-point of $\Lambda^1_{n_1,e_1}$.

We conclude that $\Lambda := \Lambda_{m_0} \cap \Lambda^0_{n_0,e_0} \cap \Lambda^1_{n_1,e_1}$ is a locally closed subset in $X_{\infty\infty}$, that Φ is a K-point of Λ and that any L-point Ψ of Λ is a L-wedge on X, which does not lift to Y, since $\Psi^\sharp(g_1)/\Psi^\sharp(g_0) \notin L[[\xi, t]]$.

In case (ii), let $n_2 := inf\{n \in \mathbb{N} \ / \ e(n) < (n+1)e_0\}$. The L-wedges Ψ such that

$ord_t \Psi^\sharp(g_1) \geq n_0$,

$ord_\xi P_n \left((G_0^\Psi)_{n_0}, (G_1^\Psi)_{n_0}, \ldots, (G_0^\Psi)_{n_0+n}, (G_1^\Psi)_{n_0+n}\right) \geq (n+1)e_0$, for $0 \leq n < n_2$,

$ord_\xi P_{n_2} \left((G_0^\Psi)_{n_0}, (G_1^\Psi)_{n_0}, \ldots, (G_0^\Psi)_{n_0+n_2}, (G_1^\Psi)_{n_0+n_2}\right) = e(n_2)$

are the L-points of a locally closed subset $\Lambda^{0,1}_{n_0,e_0,e(n_2)}$ in $X_{\infty\infty}$ and Φ is a K-point of $\Lambda^{0,1}_{n_0,e_0,e(n_2)}$.

We conclude that $\Lambda := \Lambda_{m_0} \cap \Lambda^0_{n_0,e_0} \cap \Lambda^{0,1}_{n_0,e_0,e(n_2)}$ is a locally closed subset in $X_{\infty\infty}$, that Φ is a K-point of Λ, and that any L-point Ψ of Λ is a L-wedge on X which does not lift to Y, since again $\Psi^\sharp(g_1)/\Psi^\sharp(g_0) \notin L[[\xi, t]]$.

Remark 2.6. A similar statement holds for wedges which lift to Y, but we do not need it in the sequel.

The following lemma will be crucial in the proof of proposition 2.9.

Lemma 2.7. *Let A be a noetherian integral ring and let B be a countably generated A-algebra. Let $\rho : Z = Spec\ B \to S = Spec\ A$ be the induced morphism of affine schemes. If Λ is a locally closed subset of Z such that the generic point of S lies in the image $\rho(\Lambda)$ of Λ in S, then there exists a countable family of dense open subsets $\{U_n\}_{n\in\mathbb{N}}$ of S such that $\rho(\Lambda) \supseteq \cap_{n\in\mathbb{N}} U_n$.*

Proof : By assumption $\Lambda = U \cap F$ where U is an open set and F is a closed set in Z. We have $F = \text{Spec}\ B/I$ for an ideal I in B, so B/I is a countably generated A-algebra. Now Λ is an open set in F and $\rho'(\Lambda) = \rho(\Lambda)$ where $\rho' : F \hookrightarrow Z \to S$. This reduces to prove the lemma for an open subset Λ of Z.

The open sets of the form $D(g)$, for $g \in B$, complement of the closed set $V((g))$, form a base of the Zariski topology of $Z = \text{Spec}\ B$. So there exist $g_i \in B$, $i \in I$, such that $\Lambda = \cup_{i\in I} D(g_i)$. We have $\rho(\Lambda) = \cup_{i\in I}\rho(D(g_i))$, so there exists $i \in I$ such that the generic point of S lies in $\rho(D(g_i))$. This reduces to prove the lemma for an open subset Λ of Z of the form $D(g)$. But $D(g) = \text{Spec}\ B_g$ where B_g is the localized ring, which is isomorphic to $B[x]/(gx-1)$, hence a countably generated A-algebra. This reduces to prove the lemma for $\Lambda = Z$.

We have $B = \cup_{n\in\mathbb{N}} B_n$ where B_n is a finitely generated A-algebra contained in B. Set $Z_n = \text{Spec}\ B_n$ and let $\sigma_n : Z \to Z_n$ and $\rho_n : Z_n \to S$ denote the canonical maps. For $n \in \mathbb{N}$, we have $\rho = \rho_n \circ \sigma_n$, so $\rho(Z) \subseteq \cap_n \rho_n(Z_n)$. Now let $\wp$ be a point of Spec A which does not lie in $\rho(Z)$. The fiber of ρ over this point is empty, that is $\wp B_\wp = B_\wp$. So there exist an integer $e \geq 1$, $a_i \in \wp$, $b_i \in B$, $1 \leq i \leq e$, and $s \in A \setminus \wp$, such that $1 = \sum_{1\leq i\leq e} a_i\ b_i/s$ in $B_\wp$, i.e. there exists $s' \in A \setminus \wp$ such that $s'\ (s - \sum_{1\leq i\leq e} a_i\ b_i) = 0$. Now, there exists n_0 such that $b_i \in B_{n_0}$ for $1 \leq i \leq e$.

Hence $\wp(B_{n_0})_\wp = (B_{n_0})_\wp$, or equivalently $\wp$ does not lie in $\rho_{n_0}(Z_{n_0})$. We conclude that $\rho(Z) = \cap_n \rho_n(Z_n)$. But, by Chevalley's theorem, each $\rho_n(Z_n)$ is constructible in S. By assumption, the generic point of S lies in $\rho(Z)$, hence in $\rho_n(Z_n)$ for every n, so $\rho_n(Z_n)$ contains a dense open subset U_n of S and $\rho(Z)$ contains $\cap_n U_n$.

Definition 2.8. *([Re3], definition 5.1) Let $p : Y \to X$ be a resolution of singularities of X, E be an exceptional divisor on Y, and let P_E be the generic point of the irreducible closed subset N_E of X_∞^{Sing} defined in 2.2.*

We say that p satisfies the property of lifting wedges centered at P_E if, for any field extension K of the residue field $\kappa(P_E)$ of P_E on X_∞, any K-wedge on X whose special arc is P_E, lifts to Y.

We say that p satisfies the property of lifting wedges in $X_{\infty,\infty}^{Sing}$ centered at P_E if, for any field extension K of $\kappa(P_E)$, any K-wedge on X whose special arc is P_E and whose generic arc belongs to X_∞^{Sing}, lifts to Y.

Note that the property of lifting wedges centered at P_E is stronger that the property of lifting wedges in $X_{\infty,\infty}^{\mathrm{Sing}}$ centered at P_E.

Proposition 2.9. *Let $p : Y \to X$ be an Hironaka resolution of singularities of X, let E be an irreducible component of the exceptional locus of p, and let U be an open subset of E as in lemma 2.3. Given a point $Q_0 \in U$, we set $Y_\infty^\dagger(Q_0) = Y_\infty^\dagger(U) \cap (j_0^Y)^{-1}(Q_0)$, and $N^\dagger(Q_0) = p_\infty(Y_\infty^\dagger(Q_0))$. Suppose that the following property holds :*

> *"For every countable family $\{U_n\}_{n\in\mathbb{N}}$ of nonempty open subsets of U, there exists a closed point Q_0 in $\cap_n U_n$ such that, for any closed point $\mathbf{P}$ of $X_{\infty,\infty}$ (resp. $X_{\infty,\infty}^{Sing}$) in $j_{0,\infty}^{-1}(N^\dagger(Q_0))$ (i.e. such that the special arc of the corresponding wedge $\Phi_{\mathbf{P}}$ on X lies in $N^\dagger(Q_0)$), the wedge $\Phi_{\mathbf{P}}$ lifts to Y."*

Then, p satisfies the property of lifting wedges centered at P_E (resp. p satisfies the property of lifting wedges in $X_{\infty,\infty}^{Sing}$ centered at P_E).

Proof : Once again, we may assume X to be affine. First note that, since p is an Hironaka resolution of singularities, E has codimension 1 on Y. Suppose that p does not satisfy the property of lifting wedges centered at P_E (resp. p does not satisfy the property of lifting wedges in $X_{\infty,\infty}^{\mathrm{Sing}}$ centered at P_E). Then, there exist a field extension K of $\kappa(P_E)$, and a K-wedge Φ on X (resp. a K-wedge Φ whose generic arc belongs to X_∞^{Sing}) whose special arc is P_E which does not lift to Y. By lemma 2.5 there exists a locally closed subset Λ of $X_{\infty,\infty}$ such that Φ is a K-point of Λ and such that, for any field extension $k \subset L$, any L-point of Λ is a L-wedge on X which does not lift to Y. Let Ω_0 be the affine open subset of Y and let U be the open subset of $\Omega_0 \cap E$ defined in the proof of lemma 2.3. Since p_∞ induces an isomorphism of schemes $Y_\infty^\dagger(U) \to N^\dagger(U)$, the map

$$\rho : j_{0,\infty}^{-1}(N^\dagger(U)) \to N^\dagger(U) \xrightarrow{\sim} Y_\infty^\dagger(U) \to U \subset \Omega_0 \cap E$$

induced by the projections $j_{0,\infty} : X_{\infty,\infty} \to X_\infty$, and $j_0^Y : Y_\infty \to Y$ is a morphism of schemes. Now $\mathcal{O}(X_{\infty,\infty})$ is a countably generated $\mathcal{O}(X_\infty)$-algebra via $j_{0,\infty}$ and, setting $Y_\infty^{\Omega_0\cap E} := (j_0^Y)^{-1}(\Omega_0 \cap E)$, $\mathcal{O}(Y_\infty^{\Omega_0\cap E})$ is a countably generated $\mathcal{O}(\Omega_0 \cap E)$-algebra via j_0^Y. Moreover $Y_\infty^\dagger(U) = Y_\infty^{\Omega_0\cap E} \setminus V((G_0)_{n_0})$ where, as in lemma 2.3, p is the blowing-up of the ideal I of $\mathcal{O}(X)$ and $I\mathcal{O}(\Omega_0) = (g_0)\mathcal{O}(\Omega_0)$. So, the morphism ρ enjoys the hypothesis of lemma 2.7. Let $\Lambda' := \Lambda \cap j_{0,\infty}^{-1}(N^\dagger(U))$ (resp. $:= \Lambda \cap j_{0,\infty}^{-1}(N^\dagger(U)) \cap X_{\infty,\infty}^{\mathrm{Sing}}$), which is a locally closed subset of $X_{\infty,\infty}$. Moreover since Φ is a K-point of Λ' and the special arc of Φ is P_E, its image in Λ' is a point of Λ' which is mapped to the generic point of $\Omega_0 \cap E$ by ρ. Therefore, by lemma 2.7,

there exists a countable family of nonempty open subsets $\{U_n\}_{n\in\mathbb{N}}$ of U such that $\cap_n U_n \subset \rho(\Lambda')$. Since k is uncountable, $\cap_n U_n$ has closed points. For any such closed point $Q_0 \in \cap_n U_n$, $\rho^{-1}(Q_0) \cap \Lambda'$ is a nonempty locally closed subset of $X_{\infty,\infty}$. So the wedge $\Phi_{\mathbf{P}}$ corresponding to any closed point $\mathbf{P} \in \rho^{-1}(Q_0) \cap \Lambda'$ does not lift to Y. This contradicts the hypothesis.

The above proposition suggests to introduce :

Definition 2.10. *Let $p : Y \to X$ be a resolution of singularities of X, and let E be an exceptional divisor on Y. We say that p satisfies the property of lifting wedges (resp. p satisfies the property of lifting wedges in $X^{Sing}_{\infty,\infty}$)) with respect to E, if the set of closed points Q_0 in E such that every wedge $\Phi_{\mathbf{P}}$ given by a closed point $\mathbf{P}$ of $X_{\infty,\infty}$ (resp. $X^{Sing}_{\infty,\infty}$) whose special arc lies in $N^{\dagger}(Q_0)$ lifts to Y, has a nonempty intersection with $\cap_{n\in\mathbb{N}} U_n$ for every family of dense open subsets $\{U_n\}_{n\in\mathbb{N}}$ of E.*

2.11. An *essential divisor* over X is a divisorial valuation ν of the function field $k(X)$ of X centered in Sing X, such that the center of ν on any resolution of singularities $p : Y \to X$ is an irreducible component of the exceptional locus of p ([Na] p. 35, and [IK]). Given a resolution of singularities $p : Y \to X$, and an exceptional divisor E on Y, if the divisorial valuation ν defined by E is essential, then the point P_E of X_∞ defined in 2.2 only depends on ν, not on Y. Let $\{\nu_\alpha\}_{\alpha\in\mathcal{E}}$ be the set of essential divisors over X, and for each $\alpha \in \mathcal{E}$, set $P_\alpha := P_{E_\alpha}$, where E_α is the center of ν_α in some divisorial resolution. Let $N_\alpha := \overline{\{P_\alpha\}}$, then $\nu_\alpha := \nu_{P_\alpha}$ only depends on N_α.

Since char $k = 0$, we have

$$X_\infty^{\mathrm{Sing}} = \bigcup_{\alpha\in\mathcal{E}} N_\alpha$$

([Na], [IK], see also [Re2] prop. 2.2).

Definition 2.12. *([Na]) We call the map*

$$\mathcal{N}_X : \quad \{\textit{irreducible components of } X_\infty^{Sing}\} \longrightarrow \{\textit{essential divisors over } X\}$$

sending $N_\alpha \mapsto \nu_\alpha$ the "Nash map".

The map $\mathcal{N}_X$ is injective but it need not be surjective if $\dim X \geq 4$ ([IK]). Following [Le], we are looking for a characterization of the image of $\mathcal{N}_X$ involving wedges. The first approach appears in [Re2] :

Proposition 2.13. *([Re2], theorem 5.1) Let ν_α be an essential divisor over X, and let $\kappa(P_\alpha)$ be the residue field of P_α in X_∞. The following conditions are equivalent :*

(i) ν_α belongs to the image of the Nash map $\mathcal{N}_X$.

(ii) For any resolution of singularities $p : Y \to X$, p satisfies the property of lifting wedges in $X^{Sing}_{\infty,\infty}$ centered at P_α, i.e. for any field extension K of $\kappa(P_\alpha)$, any K-wedge on X whose special arc is P_α and whose generic arc belongs to X_∞^{Sing}, lifts to Y.

(ii') There exists a resolution of singularities $p : Y \to X$ satisfying the property of lifting wedges in $X^{Sing}_{\infty,\infty}$ centered at P_α.

The previous result follows from the theory of *stable* points of X_∞ developed in [Re2] (see also [Re3]), whose key is to understand the finiteness properties of the

points P_E defined in 2.2. The main result in [Re2] is the following :

2.14. Finiteness property of the stable point P_E *([Re2], th. 4.1)*. The maximal ideal $P_E\ \mathcal{O}_{(X_\infty)_{\mathrm{red}},P_E}$ of the local ring $\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_E}$ is finitely generated. Therefore, $\widehat{\mathcal{O}_{(X_\infty)_{\mathrm{red}},P_E}}$ is a Noetherian complete local ring.

Taking into account that, for an uncountable algebraically closed field k, the closed points of $X_{\infty,\infty}$ are the k-rational points of $X_{\infty,\infty}$ (see [Is] prop. 2.10), combining propositions 2.9 and 2.13, we conclude :

Corollary 2.15. *Let k be an uncountable algebraically closed field of characteristic zero. Let ν_α be an essential divisor over X, and let $p : Y \to X$ be an Hironaka resolution of singularities. Assume that the set of k-points Q_0 in E_α such that every k-wedge Φ whose special arc lies in $N^\dagger(Q_0)$ and whose generic arc belongs to X_∞^{Sing} lifts to Y has a nonempty intersection with $\cap_{n\in\mathbb{N}} U_n$ for every family of dense open subsets $\{U_n\}_{n\in\mathbb{N}}$ of E. Then ν_α belongs to the image of the Nash map $\mathcal{N}_X$.*

3. Exceptional divisors which are not uniruled

In this section k is assumed to be an uncountable algebraically closed field of characteristic zero.

Definition 3.1. *Let E be a variety over k of dimension d. We say that E is uniruled if there exists a variety E' over k of dimension $d-1$, and a dominant rational map $\mathbb{P}^1_k \times E' \cdots \to E$.*

Lemma 3.2. *Let E be an irreducible variety over k which is projective and nonsingular. Then E is uniruled if and only if there exists a countable family $\{U_n\}_{n\in\mathbb{N}}$ of nonempty open subsets of E such that, for every k-point Q_0 in $\cap_n U_n$, there is a k-rational curve in E going through Q_0.*

Proof : If E is uniruled, there is a k-rational curve through every k-point of E ([Ko] chap.IV, cor. 1.4.4, [De] chap. 4 rem. 4.2 (4)). Conversely, since E is projective and nonsingular, it is enough to show that E contains a k-rational curve C over which the tangent bundle T_E is generated by global sections. Such a curve is said to be free ([Ko] chap. IV th. 1.9, [De] chap. 4, cor. 4.11). Now, since char $k = 0$, by [Ko] chap. II th. 3.11, [De] chap. 4, prop. 4.14, there exists a subset E^{free} of E and dense open subsets V_n, $n \in \mathbb{N}$, of E such that $E^{\mathrm{free}} = \cap_n V_n$ and such that any k-rational curve D with a nonempty intersection with E^{free} is free. Let Q_0 be a k-rational point of E in $(\cap_n U_n) \cap (\cap_n V_n)$. Such a point exists because k is uncountable and algebraically closed. By assumption, there is a k-rational curve going through Q_0, hence it is free and E is uniruled.

Theorem 3.3. *Let X be a variety over an uncountable algebraically closed field of characteristic zero k. Let $p : Y \to X$ be a resolution of singularities of X, and let E be an exceptional divisor on Y.*

If E is not uniruled, then the divisorial valuation defined by E is an essential divisor which belongs to the image of the Nash map.

Proof : By [Ab1] prop. 4, if the valuation ν defined by E is not an essential divisor, then E is birationally ruled, i.e. E is birationally isomorphic to $F \times \mathbb{P}^1_k$ for some k-variety F. In particular, E is uniruled. So ν is an essential divisor. Hence,

we may assume that $p : Y \to X$ is an Hironaka resolution of singularities with nonsingular exceptional divisors, since the center of ν on any such resolution is a divisor birationally equivalent to E, hence not uniruled. Therefore, in view of prop. 2.13 (ii'), we only have to prove that p satisfies the property of lifting wedges in $X^{\mathrm{Sing}}_{\infty,\infty}$ centered at P_E. To do so, we will apply cor. 2.15.

Let U be an open subset of E as in lemma 2.3, and let $\{U_n\}_{n\in\mathbb{N}}$ be a countable family of nonempty open subsets of U. By lemma 3.2, since E is nonsingular and not uniruled, there exists a k-point Q_0 in $\cap_n U_n$, such that no k-rational curve in E goes through Q_0. Let Φ be a k-wedge on X whose special arc is in $N^{\dagger}(Q_0)$. We will prove that Φ lifts to Y.

Let $Z_0 := \mathrm{Spec}\ k[[\xi, t]]$. The wedge $\Phi : Z_0 \to X$ does not factor through Sing X, because the special arc of Φ does not. Hence there exists a finite sequence of point blowing-ups $q : Z \to Z_0$ which eliminates the points of indeterminacy of the rational map $Z_0 \dashrightarrow Y$ induced by Φ, so we have a commutative diagram of k-morphisms

$$\begin{array}{ccc} Z & \xrightarrow{\Phi'} & Y \\ q\downarrow & & \downarrow p \\ Z_0 & \xrightarrow{\Phi} & X. \end{array}$$

Let C be the exceptional locus of q. Since C is connected, its image $\Phi'(C)$ by $\Phi' : Z \to Y$ is a connected subset of the exceptional locus of p which contains Q_0. Since C is a finite union of $\mathbb{P}^1_k$'s, by Lüroth's theorem, our hypothesis on E implies that $\Phi'(C)$ is the point Q_0. Thus Φ' factors through Spec $\mathcal{O}_{Y,Q_0}$, i.e. we have

$$\Phi' : Z \longrightarrow \mathrm{Spec}\ \mathcal{O}_{Y,Q_0} \ \subset\ Y.$$

Indeed Φ' factors through every open subset V of Y containing Q_0, or equivalently $Z = \Phi'^{-1}(V)$. This is because $Z \setminus \Phi'^{-1}(V)$ is a closed subset of Z contained in $Z \setminus C$, hence q induces an isomorphism onto its image F in Z_0. Thus F is a closed subset of Z_0 which does not contain the closed point $O = q(C)$ of Z_0. This implies that $F = \emptyset$, so $Z = \Phi'^{-1}(V)$.

Finally, since Z_0 is normal, we have that $q_*\mathcal{O}_Z \cong \mathcal{O}_{Z_0}$ hence an isomorphism $k[[\xi, t]] \cong \mathcal{O}(Z)$. So, for every $y \in \mathcal{O}_{Y,Q_0}$, its image in $\mathcal{O}(Z)$ lies in $k[[\xi, t]]$, i.e. there exists a morphism $\widetilde{\Phi} : Z_0 \to \mathrm{Spec}\ \mathcal{O}_{Y,Q_0}$ such that $\widetilde{\Phi} \circ q = \Phi'$. Since the morphism q is birational, we get that $p \circ \widetilde{\Phi} = \Phi$. i.e. Φ lifts to Y.

4. Appendix

Combining Luröth's and Grauert's contraction theorems, we show in this appendix that a positive answer to the Nash problem for normal surface singularities over $\mathbb{C}$ would follow from proving the property of lifting wedges with respect to the essential divisors over those surface singularities over $\mathbb{C}$ which are quasirational.

Definition 4.1. *([Ab2]) Let S be a surface over an algebraically closed field k (i.e. a variety of dimension 2 over k), and let P be a singular point of S at which S is normal. We say that S has a quasirational singularity at P if there exists a resolution of singularities $p : Y \to (S, P) :=$ Spec $\mathcal{O}_{S,P}$ such that the irreducible components of the exceptional locus $p^{-1}(P)$ are rational curves.*

Note that the same holds for every resolution of singularities of (S, P). Also note that, since the inverse image by the map $\widehat{(S, P)} :=$ Spec $\widehat{\mathcal{O}_{S,P}} \to (S, P)$ of a resolution of singularites of (S, P) is a resolution of singularites of $\widehat{(S, P)}$ ([Li] lemma 16.1),

this is equivalent to saying that there exists a resolution of singularities of $\widehat{(S,P)}$ (or equivalently for all) such that the irreducible components of the exceptional locus are rational curves. Here $\widehat{\mathcal{O}_{S,P}}$ is the completion of $\mathcal{O}_{S,P}$ for the $M_{S,P}$-adic topology.

Proposition 4.2. *If every quasirational surface singularity over $\mathbb{C}$ has a resolution of singularities which satisfies the property of lifting wedges with respect to each exceptional essential divisor, then, for every normal surface S over $\mathbb{C}$, the Nash map $\mathcal{N}_S$ is surjective.*

Proof : We may assume S to be affine. Let ν be an essential divisor over S, let $p : Y \to S$ be an Hironaka resolution of singularities of S, and let E be the exceptional curve, center of ν on Y. If E is not a rational curve, we already know by theorem 3.3 that ν belongs to the image of the Nash map. If not, let $P \in S$ denote the isolated singular point of S such that $p(E) = P$, let $\{E_\beta\}_{\beta \in B}$ be the irreducible components of $p^{-1}(P)$ which are rational curves and let $\mathbb{E}$ be the connected component of $\cup_{\beta \in B} E_\beta$ which contains E. The intersection matrix of the irreducible curves in $\mathbb{E}$ is negative definite. Hence by Grauert's contraction theorem ([Gr] p. 367), there exists an analytic normal surface X and a proper holomorphic map $p_1 : Y^{\mathrm{an}} \to X$ which contracts $\mathbb{E}$ to a point $Q \in X$ and induces a biholomorphic map from $Y^{\mathrm{an}} \setminus \mathbb{E}$ to $X \setminus Q$. Here Y^{an}, below S^{an}, denotes the associated complex analytic space. The holomorphic map $X \setminus Q \cong Y^{\mathrm{an}} \setminus \mathbb{E} \hookrightarrow Y^{\mathrm{an}} \to S^{\mathrm{an}}$ extends to a holomorphic map $p_2 : X \to S^{\mathrm{an}}$. Indeed, since S is normal, S^{an} is again normal, so for any open neighborhood V of P on S^{an} (in the complex topology) we have an isomorphism $\mathcal{O}_{S^{\mathrm{an}}}(V) \xrightarrow{\sim} \mathcal{O}_{Y^{\mathrm{an}}}(p^{-1}(V))$. Now $V_1 = p_1(p^{-1}(V))$ is an open neighborhood of Q on X and $p_1^{-1}(V_1) = p^{-1}(V)$, and, X being normal, we also have an isomorphism $\mathcal{O}_X(V_1) \xrightarrow{\sim} \mathcal{O}_{Y^{\mathrm{an}}}(p_1^{-1}(V_1))$. The natural isomorphisms $\mathcal{O}_{S^{\mathrm{an}}}(V) \cong \mathcal{O}_X(V_1)$ so obtained give rise to a local morphism $\mathcal{O}_{S^{\mathrm{an}},P} \to \mathcal{O}_{X,Q}$, hence to a holomorphic map $p_2 : X \to S^{\mathrm{an}}$ such that $p = p_2 \circ p_1$.

Let now $\Phi : Z_0 := \mathrm{Spec}\ \mathbb{C}[[\xi, t]] \to S$ be a $\mathbb{C}$-wedge on S, whose special arc lies in $N^\dagger(Q_0)$ with Q_0 a $\mathbb{C}$-point of E in the nonsingular locus of $p^{-1}(P)_{\mathrm{red}}$. We will first show that Φ lifts to X. As in the proof of theorem 3.3, let $q : Z \to Z_0$ be the finite sequence of point blowing-ups which eliminates the points of indeterminacy of the rational map $Z_0 \cdots \to Y$ induced by Φ, let C be the exceptional locus of q and let

$$\begin{array}{ccc} Z & \xrightarrow{\Phi'} & Y \\ q \downarrow & & \downarrow p \\ Z_0 & \xrightarrow{\Phi} & S \end{array}$$

be the resulting commutative diagram. Here again $\Phi'(C)$ is a connected subset of $p^{-1}(P)$ which has a nonempty intersection with E, so by Luröth's theorem, we have $\Phi'(C) \subset \mathbb{E}$.

Since $p_1^{-1}(Q)$ is a compact analytic subspace of $p^{-1}(P)^{\mathrm{an}}$ with reduced subspace $\mathbb{E}^{\mathrm{an}}$, and $p^{-1}(P)$ is a projective scheme, by GAGA ([Se]) there exists an ideal sheaf $\mathcal{I}$ on Y such that $M_{X,Q}\mathcal{O}_{Y^{\mathrm{an}}} = \mathcal{I}\mathcal{O}_{Y^{\mathrm{an}}}$, and for every $n \geq 1$, we have $\Gamma(Y^{\mathrm{an}}, \mathcal{O}_{Y^{\mathrm{an}}}/\mathcal{I}^n\mathcal{O}_{Y^{\mathrm{an}}}) = \Gamma(Y, \mathcal{O}_Y/\mathcal{I}^n)$. Besides, by Grauert's comparison theorem ([BS] chap. III sec.3), the natural map

$$(p_1)_*(\mathcal{O}_{Y^{\mathrm{an}}})_Q \otimes_{\mathcal{O}_{X,Q}} \widehat{\mathcal{O}_{X,Q}} \to \varprojlim \Gamma(Y^{\mathrm{an}}, \mathcal{O}_{Y^{\mathrm{an}}}/\mathcal{I}^n\mathcal{O}_{Y^{\mathrm{an}}})$$

is an isomorphism. But X being normal, we have $(p_1)_*(\mathcal{O}_{Y^{\mathrm{an}}}) = \mathcal{O}_X$, so we get a natural isomorphism

$$\widehat{\mathcal{O}_{X,Q}} \xrightarrow{\sim} \varprojlim \Gamma(Y, \mathcal{O}_Y/\mathcal{I}^n) = \Gamma(Y, \varprojlim \mathcal{O}_Y/\mathcal{I}^n) =: \Gamma(\widehat{Y}_{\mathbb{E}}, \mathcal{O}_{\widehat{Y}_{\mathbb{E}}})$$

Here $\widehat{\mathcal{O}_{X,Q}}$ is the completion of $\mathcal{O}_{X,Q}$ for the $M_{X,Q}$-adic topology and $\widehat{Y}_{\mathbb{E}}$ is the formal completion of Y along $\mathbb{E}$.

Now, since $C \subset \Phi'^{-1}(\mathbb{E})$, there exists an effective divisor D whose support is C and an ideal sheaf $\mathcal{J}$ on Z such that $\mathcal{I}\mathcal{O}_Z = \mathcal{J}\mathcal{O}_Z(-D)$. For every $n \geq 1$, we thus get natural morphisms

$$\Gamma(Y, \mathcal{O}_Y/\mathcal{I}^n) \to \Gamma(Z, \mathcal{O}_Z/\mathcal{I}^n\mathcal{O}_Z) \to \Gamma(Z, \mathcal{O}_Z/\mathcal{O}_Z(-nD))$$

from which we derive a natural morphism $\widehat{\mathcal{O}_{X,Q}} \to \lim_{\leftarrow} \Gamma(Z, \mathcal{O}_Z/\mathcal{O}_Z(-nD)) =: \Gamma(\widehat{Z}_C, \mathcal{O}_{\widehat{Z}_C})$ where $\widehat{Z}_C$ is the formal completion of Z along C. But C is the support of both $\mathcal{O}_Z/M_{Z_0,O}\mathcal{O}_Z$ and $\mathcal{O}_Z/\mathcal{O}_Z(-D)$, so we have $\Gamma(\widehat{Z}_C, \mathcal{O}_{\widehat{Z}_C}) = \lim_{\leftarrow} \Gamma(Z, \mathcal{O}_Z/M^n_{Z_0,O}\mathcal{O}_Z)$, and since $q_*\mathcal{O}_Z = \mathcal{O}_{Z_0}$, the natural map $\widehat{\mathcal{O}_{Z_0,O}} \to \lim_{\leftarrow} \Gamma(Z, \mathcal{O}_Z/M^n_{Z_0,O}\mathcal{O}_Z)$ is an isomorphism by the theorem on formal functions ([Ha]). So we get a natural map $\widehat{\mathcal{O}_{X,Q}} \to \widehat{\mathcal{O}_{Z_0,O}}$, hence a $\mathbb{C}$-wedge $\Psi : Z_0 \to \widehat{(X,Q)} := \mathrm{Spec}\ \widehat{\mathcal{O}_{X,Q}}$.

Finally, we have $p_2 \circ \Psi = \Phi$, where here, for simplicity, we have denoted Φ (resp. p_2) the morphism $Z_0 \to \widehat{(S,P)} := \mathrm{Spec}\ \widehat{\mathcal{O}_{S,P}}$ (resp. $\widehat{(X,Q)} \to \widehat{(S,P)}$) induced by Φ (resp. p_2). Indeed, once again by the theorem on formal functions, the natural map $\widehat{\mathcal{O}_{S,P}} \to \lim_{\leftarrow} \Gamma(Y, \mathcal{O}_Y/M^n_{S,P}\mathcal{O}_Y) =: \Gamma(\widehat{Y}_{p^{-1}(P)}, \mathcal{O}_{\widehat{Y},p^{-1}(P)})$ is an isomorphism, and the diagram

$$\begin{tikzcd} & \widehat{Y}_{\mathbb{E}} \arrow[dr] & \\ \widehat{Z}_C \arrow[ur] \arrow[rr] & & \widehat{Y}_{p^{-1}(P)} \end{tikzcd}$$

is commutative.

Now, since the surface X has an isolated singularity at Q, there exists an (algebraic) surface S_1 over $\mathbb{C}$ and a point P_1 of S_1 such that $\widehat{\mathcal{O}_{S_1,P_1}}$ and $\widehat{\mathcal{O}_{X,Q}}$ are $\mathbb{C}$-isomorphic ([Ar] th. 3.8). Moreover S_1 has a quasirational singularity at P_1. To prove that, it is enough to check that Q is a singular point of X, and that there exists a resolution of singularities of $\widehat{(X,Q)}$ such that the irreducible components of its exceptional locus are rational curves. But, since E is the center on Y of an essential divisor over S, its image on the minimal resolution of singularities of X is again a curve, hence Q is a singular point of X.

Besides let $\widetilde{Y}$ denote the schematic closure of the inverse image of $\widehat{(X,Q)} \backslash p_2^{-1}(P)$ in $Y \times_S \widehat{(X,Q)}$ and let $\widehat{p_1} : \widetilde{Y} \to \widehat{(X,Q)}$ denote the natural projection. The map $\widehat{p_1}$ is a resolution of singularities of $\widehat{(X,Q)}$ and its exceptional locus $\widehat{p_1}^{-1}(Q)$ is $\mathbb{C}$-isomorphic to $\mathbb{E}$, hence its irreducible components are rational curves. This also shows that every resolution of singularities $\pi_1 : Y_1 \to S_1$ has an exceptional essential divisor E_1 birationally equivalent to E. But, if π_1 satisfies the property of lifting wedges with respect to E_1, the set of $\mathbb{C}$-points Q_0 in the intersection of a dense open set of E isomorphic to a dense open set of E_1 with the nonsingular locus of

$p^{-1}(P)_{\mathrm{red}}$, such that every $\mathbb{C}$-wedge Φ whose special arc lies in $N^{\dagger}(Q_0)$ lifts to Y, has a nonempty intersection with $\cap_{n\in\mathbb{N}}U_n$ for every family of dense open subsets $\{U_n\}_{n\in\mathbb{N}}$ of E. By corollary 2.15, the essential divisor $\nu = \nu_E$ over S belongs to the image of the Nash map $\mathcal{N}_S$.

Remark 4.3. Note that the generic arc of the lifting Ψ to $\widehat{(X,Q)}$ of a $\mathbb{C}$-wedge Φ on $\widehat{(S,P)}$ whose generic arc is centered at P need not be centered at Q. Therefore proving that every quasirational surface singularity (S_1, P_1) has a resolution of singularities which satisfies the property of lifting wedges in ${S_1}^{\mathrm{Sing}}_{\infty,\infty}$ with respect to each essential divisor would not be enough to insure the surjectivity of the Nash map for every normal surface over $\mathbb{C}$.

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